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Global worldsheet anomalies from M-theory

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Abstract: We present an M-theory proof of the anomaly of Freed and Witten which in general shifts the quantisation law for the U(1) gauge field on a D6-brane. The derivation requires an understanding of how fields on the D6-brane lift to M-theory, together with a localisation formula which we prove using a U(1)-index theorem. We also show how the anomaly is related to the K-theory classification of Ramond-Ramond fields. In addition we discuss the M-theory origin of the D6-brane effective action, and illustrate the general arguments with a concrete example.

Keywords: M-Theory, D-branes, Anomalies in Field and String Theories.

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1. Introduction and summary

In recent years D6-branes have proved to be a valuable tool for understanding various aspects of M-theory. For example, one can construct semi-realistic Standard-like Models in four dimensions from configurations of D6-branes — for a recent example, see [1] and references therein. Such configurations are also closely related to the subject of M-theory on conical singularities, where again the dual description in terms of D6-branes in type IIA has proved useful in understanding the dynamics — see, for example, [2]–[4]. From the M-theory perspective, the D6-brane is a Kaluza-Klein monopole. Reduction from M-theory to type IIA involves choosing an "M-theory circle", and the D6-brane is then, roughly speaking, a codimension four locus Q over which this circle degenerates. In this way, M-theory is dual to type IIA string theory with D6-branes wrapped on Q. The dynamics of the latter may then often be understood using standard string theory techniques.

The low energy dynamics of a D6-brane, as for all D-branes, is governed by a Born-Infeld effection action, together with certain Wess-Zumino terms which couple the world-volume fields to the type IIA Ramond-Ramond fields. For the simple case of a flat D6-brane linearly embedded in flat Minkowski spacetime, it is fairly straightforward to derive (much of) this effective action from Kaluza-Klein reduction of eleven-dimensional supergravity on Taub-NUT space [5] (for work on the dyonic nature of this D6-brane, see [6, 7]). However, in general it seems that the precise way in which the D6-brane dynamics arises from

M-theory is not well-understood. This gap was partially filled in [8] where it was shown how the gravitational Wess-Zumino terms on a D6-brane arise from the M-theory effective action — for M-theory compactified on a spin eight-manifold, the gravitational couplings on a D6-brane were shown to arise from a non-standard Kaluza-Klein reduction of a higher-derivative gravitational correction to the eleven-dimensional supergravity action, in which bulk couplings reduce to brane couplings.

In this note we address what is perhaps an even more fundamental question: how does the gauge field on the D6-brane arise from M-theory? Recall that on every D-brane there propagates a U(1) gauge field A, with field strength $F = \mathrm{d}A$. If we measure the gauge field in terms of the flux of its field strength, $[F] \in H^2(Q;\mathbb{R})$, where the brackets denote cohomology class, then we would like to know how $[F]$ is related to the dual M-theory description. Moreover, the field strength F is Dirac quantised in the quantum theory, and one should again be able to see this from M-theory. In fact, there is an interesting subtlety in this quantisation condition which will be a focal point of this note. Naively, one expects the periods of F to be integer multiples of 2π, so that in fact $[F/2\pi] \in H^2(Q;\mathbb{Z})$ — this is Dirac quantisation. However, by studying global worldsheet anomalies for a fundamental string which ends on a D-brane Q, Freed and Witten [9] showed that the gauge field strength F on a D-brane should in general satisfy a modified form of Dirac quantisation

$$\int_U \frac{F}{2\pi} = \frac{1}{2}\int_U w_2(Q) \mod \mathbb{Z}\,, \tag{1.1}$$

where $U \subset Q$ is any two-cycle on the D-brane Q, and $w_2(Q)$ is the second Stiefel-Whitney class of Q. The latter is non-zero precisely when Q is not a spin manifold. Thus when spinors exist globally on Q, the quantisation condition on F is the naive one. However, more generally (1.1) says that the periods of the field strength are shifted to be half-integer multiples of 2π. In the case of a D6-brane, the question of how this shift in the quantisation of F is related to M-theory arose in a specific example in [10] where a D6-brane was wrapped on a supersymmetric cycle $\mathbb{CP}^2$. This example was subsequently analysed in considerable detail in [3, 4] — our example in section 4 is a certain compactification this.

In this paper we explain how $[F]$ is related to the dual M-theory description in the case where the M-theory background is smooth, and there is no source for the G-flux. We also show precisely how the Freed-Witten quantisation condition (1.1) arises in this context. Specifically, we begin with the membrane anomaly in M-theory and show that it reduces precisely to the Freed-Witten anomaly described above, in the particular case where the membrane reduces to a fundamental string ending on a D6-brane. The proof requires a "localisation formula" which we derive using a U(1)-index theorem. We conclude this section with a summary of our results.

In order to analyse the well-definedness of the string worldsheet path integral in the presence of D-branes, Freed and Witten studied one-parameter families of string worldsheets ending on some D-brane $Q \subset Y$, where Y denotes spacetime. Thus consider such a loop of string worldsheets $\Sigma \times \mathbf{S}^1 \subset Y$, with $U \equiv \partial\Sigma \times \mathbf{S}^1 \subset Q$, and $\mathbf{S}^1$ parametrises the loop. Since the fundamental string lifts to the membrane, we obtain a one-parameter family of membranes when we lift to M-theory. However, the D6-brane is a Kaluza-Klein monopole, and therefore each membrane W in the family must be *closed* — that is, without boundary — since there are no M5-branes or boundaries to spacetime by assumption. Thus we obtain a one-parameter family of closed membranes $V \equiv W \times \mathbf{S}^1$.

In general, the M-theory four-form field strength G also satisfies a shifted quantisation condition, which again is derived by studying one-parameter families of closed membranes [11]. The quantisation condition is

$$\int_V \frac{G}{2\pi} = \frac{1}{2}\int_V \frac{1}{16\pi^2}\operatorname{tr}\mathcal{R}\wedge\mathcal{R} \qquad \mathrm{mod}\ \mathbb{Z}\,, \tag{1.2}$$

where $\mathcal{R}$ denotes the curvature two-form for the M-theory spacetime X, and V may be any four-cycle, although the case of interest for us will be $V = W \times \mathbf{S}^1$ as defined above. If X is spin, the quantity inside the integral on the right hand side of (1.2), which is half the first Pontryagin form for X, is always an integer, but in general is not divisible by two. Thus the periods of G are shifted to be half-integer multiples of 2π in general.

The proof of the Freed-Witten anomaly proceeds in two steps. Firstly, we show that

$$\exp\left(i\int_V G\right) = \exp\left(i\int_U F\right), \tag{1.3}$$

where $U = \partial\Sigma\times\mathbf{S}^1$ is as defined above. This follows from a careful analysis of how the gauge field on the D6-brane arises from the C-field in M-theory. In particular, a crucial physical point to understand here is that, if the M-theory four-form G is everywhere smooth and closed, then there is no M5-brane charge in M-theory and thus there is no D4-brane charge in type IIA. We will see how this physical statement manifests itself mathematically in a precise way. The second part of the proof follows from a localisation formula. Specifically, we show that

$$\int_V \frac{1}{16\pi^2}\operatorname{tr}\mathcal{R}\wedge\mathcal{R} = \int_U w_2(Q) \quad \mathrm{mod}\ 2\,. \tag{1.4}$$

We prove this using a U(1)-index theorem for the Dirac operator on the membrane worldvolume. Putting (1.3) together with (1.4) therefore leads to an "M-theory derivation" of the Freed-Witten anomaly.

In section 4 we discuss the M-theory origin of the Wess-Zumino couplings on the D6-brane. The gravitational terms were treated in [8]. Here we discuss the origin of the gauge field terms.

Since some aspects of this paper are a little technical, we also include a simple concrete example in section 4. In this case, one can recover some of the results in this paper by instead analysing tadpole cancellation. This also ties in naturally with the work of [8] and the discussion of Wess-Zumino couplings. Finally, for completeness, we show in section 5 how the Freed-Witten anomaly also follows from the K-theoretic quantisation condition for the Ramond-Ramond four-form. This naturally ties in with our discussion of D4-brane charge. We conclude the paper with some speculative comments.

As a final comment in this section, notice that the only other type IIA D-brane for which the Freed-Witten anomaly may be non-trivial is the D4-brane. In this case the anomaly may be derived from M-theory by considering the partition function of the chiral two-form that propagates on the M-theory five-brane [12].

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2. Kaluza-Klein reduction

As we argued in the introduction, a string ending on a D6-brane must lift to a closed membrane worldvolume in M-theory. Our aim in the first part of this section is to describe this more accurately. In particular the discussion here will be useful in sections 3 and 5. We also give a simple example.

Consider M-theory on an oriented spin manifold X. Suppose that X comes equipped with a circle action,[1] which we will regard as rotating the "M-theory circle" direction. Thus the orbits of the group action will be the M-theory circle fibres. If $\mathrm{U}(1)$ acts freely — that is, without any fixed points — then M-theory on X is dual to type IIA string theory on the quotient space $Y = X/\,\mathrm{U}(1)$. This is usual Kaluza-Klein reduction. However, suppose now that there is a codimension four fixed point set $Q \subset X$. This is a locus on which the $\mathrm{U}(1)$ Killing vector field vanishes. Then, in this case, the quotient may still be defined, with the fixed point set being interpreted as a D6-brane in type IIA. Mathematically we are using the following local identification

$$\mathbb{R}^4/\,\mathrm{U}(1) \cong \mathbb{R}^3 \tag{2.1}$$

for the normal space to Q, where we write $\mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C}$ and the $\mathrm{U}(1)$ acts as multiplication by $e^{i\theta}$ on *both* factors, where $0 \leq \theta \leq 2\pi$ is the $\mathrm{U}(1)$ group parameter. The origin in $\mathbb{R}^3$, where the D6-brane is located, descends from the fixed origin on the left hand side of (2.1). This local model of course describes the reduction of Taub-NUT, which is topologically $\mathbb{R}^4$, to $\mathbb{R}^3$ with a Kaluza-Klein monopole at the centre. Our type IIA spacetime is then the quotient $Y = X/\,\mathrm{U}(1)$, where we use the local model (2.1) to define the quotient space near to the codimension four fixed point set. An equivalent construction may be described as follows. The circle action on X induces a complex structure on the normal bundle to Q in X. The type IIA manifold, in a neighbourhood of the D6-brane, is described by taking the projectivisation of this complex normal bundle. In more detail, for each point $p \in Q$, the normal space to Q at p is a copy of $\mathbb{C}^2$, and we simply divide out by the Hopf map $\mathbb{C}^2/\mathbb{C}_* \cong \mathbb{CP}^1$ on each space.[2] Thus the projectivisation is a $\mathbb{CP}^1 = \mathbf{S}^2$ bundle over Q. The type IIA spacetime, in a neighbourhood of Q, is thus obtained by filling in each two-sphere fibre with a three-disc. In this construction, the zero section of the disc bundle is the D6-brane worldvolume.

Consider now a string worldsheet Σ ending on a D6-brane Q, so that $\partial\Sigma \subset Q$. As we argued in the introduction, this configuration lifts to a *closed* membrane worldvolume in M-theory. For example, if $\Sigma \cong D^2$ is a two-disc, with the boundary of the two-disc $\partial\Sigma \cong \mathbf{S}^1 \subset Q$, then the M-theory lift of this worldsheet is a membrane wrapped on $W \cong \mathbf{S}^3$. To see this, it is easier to consider the inverse process of reducing from M-theory. Thus, given a closed membrane worldvolume W, we identify $W/\,\mathrm{U}(1) \cong \Sigma$ as the string worldsheet [13], where $\mathrm{U}(1)$ rotates around the M-theory circle direction. A codimension two fixed point set then naturally becomes a boundary of the worldsheet when we reduce.

[1] "circle action" and "U(1) action" will be used interchangeably.

[2] This is the same map as (2.1), except that we have, in addition, projected out the radial direction.

Locally we are using the following identification

$$\mathbb{R}^2 / \mathrm{U}(1) \cong [0, \infty)\,, \tag{2.2}$$

where 0 on the right hand side descends from the fixed origin on the left hand side. As a simple example, embed $W \cong \mathbf{S}^3$ as a sphere of unit norm in $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$, and consider the circle action which rotates the *first* factor of $\mathbb{C}$. The fixed point set is then $\{0\} \times \mathbb{C}$ which becomes a copy of $\mathbf{S}^1$ on the three-sphere. The quotient is then a two-disc D^2 where we have used the local model (2.2) on the normal space to the fixed points to yield a quotient space that has a boundary. One can make this completely explicit by writing the round metric on $\mathbf{S}^3$ as

$$\mathrm{d}s^2 = \mathrm{d}\psi^2 + \cos^2\psi \mathrm{d}\theta^2 + \sin^2\psi \mathrm{d}\phi^2\,. \tag{2.3}$$

The U(1) Killing vector we reduce on is $\partial/\partial\phi$. This vanishes at $\psi = 0$ — the locus is a circle, parameterised by θ. The quotient space is D^2 with metric

$$\mathrm{d}\psi^2 + \cos^2\psi \mathrm{d}\theta^2\,. \tag{2.4}$$

Here θ is the angular coordinate on the disc and the radial variable is $0 \le \psi \le \pi/2$, where $\psi = 0$ is the boundary of the disc and $\psi = \pi/2$ is the origin.

2.1 D4-brane charge

Consider type IIA string theory on Y, with the NS B-field temporarily set to zero. In the absence of any branes, the Bianchi identity for the Ramond-Ramond field strengths simply asserts that they are closed. In particular, G_2 and G_4 are both closed. However, in the presence of a D6-brane wrapped on Q one instead has

$$\mathrm{d}G_2 = 2\pi\delta_Q\,. \tag{2.5}$$

Equation (2.5) states that the D6-brane is a magnetic source for the M-theory Kaluza-Klein field strength, or equivalently Ramond-Ramond two-form, G_2. This follows from the fact that the circle quotient that we took is essentially a fibre-wise application of the Hopf map $\mathbf{S}^3 \to \mathbf{S}^2$, which precisely describes the Kaluza-Klein monopole. Indeed, the latter means that

$$\int_{\mathbf{S}^2} \frac{G_2}{2\pi} = 1\,, \tag{2.6}$$

where $\mathbf{S}^2$ is any two-sphere that links the D6-brane worldvolume. In (2.5) δ_Q denotes a three-form supported on Q which integrates to one over the normal space to Q. If one sets up local coordinates y_i, $i = 1, 2, 3$, on the normal space to Q, we have roughly $\delta_Q = \delta(\mathbf{y})\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3$. However, this formula is only valid locally (or in the case that the normal bundle to the brane is trivial). More generally we may use the following standard construction for δ_Q [14], which was also used in the analysis of M5-brane anomaly cancellation [15]. In a neighbourhood of Q, minus Q itself, there is always a globally-defined closed two-form e_2, known as the global angular form, which integrates to one over any two-sphere that links the D6-brane worldvolume Q — there is an explicit formula for this form in terms of the connection on the normal bundle to the brane [15, 16]. Then δ_Q may be taken to be

$$\delta_Q = \mathrm{d}\rho(r) \cdot e_2 \,, \tag{2.7}$$

where ρ is any smooth function of the radial direction r which is zero for $r \geq \epsilon$, for some $\epsilon > 0$, and is -1 near to $r = 0$. It is then easy to check that δ_Q is closed, has compact support, and integrates to 1 over the normal space to Q. However, in this construction notice that the D6-brane charge has effectively been smeared out to a radius ϵ — this is because G_2 is no longer closed inside this radius. For a truly localised D6-brane, as arises in the Kaluza-Klein reduction described above, one needs to take a limit in which $\epsilon \to 0$. In this limit, the function $\rho(r)$ simply becomes a Dirac delta-function supported at $r = 0$.

Let us now consider the Ramond-Ramond four-form, G_4. In the presence of a D6-brane, this is also not closed in general. One instead has as an equation which is of the form

$$\mathrm{d}G_4 = \delta_Q \wedge F \,. \tag{2.8}$$

Here F is the gauge field strength on the D6-brane, where a pull-back to a tubular neighbourhood of the brane is understood in (2.8). The right hand side of (2.8) arises from the Wess-Zumino couplings on the D6-brane — indeed, (2.5) also arises this way. Equation (2.8) expresses the fact that a non-zero flux of the U(1) gauge field on the D6-brane induces an effective D4-brane charge [17]. Mathematically we can interpret the right hand side of (2.8) as the cohomology class $\mathcal{T}(F)$ where $\mathcal{T} : H^*(Q) \to H^{*+3}_{\mathrm{cpt}}(Y)$ maps cohomology classes on Q to compactly supported classes in spacetime Y. For those who know about such things, $\mathcal{T}$ is essentially just the Thom isomorphism for the normal bundle NQ of Q in Y, where we identify the normal bundle with a tubular neighbourhood of Q and extend the isomorphism by zero outside this neighbourhood. More details may be found in [18] where the general case is discussed. Using Stokes' Theorem we may integrate (2.8) over the normal space to Q to obtain

$$\int_{\mathbf{S}^2} \frac{G_4}{2\pi} = \frac{F}{2\pi} \,, \tag{2.9}$$

where $\mathbf{S}^2$ is any two-sphere that links the D6-brane Q. If we have smoothed out the charge to a radius ϵ this two-sphere should have radius greater than ϵ. Since G_4 is closed away from the brane, G_4 defines a cohomology class on the complement of Q in Y. Then equation (2.9) may be regarded as a cohomological statement.

Now let us try to lift this to M-theory. Since, for non-zero flux F, we have non-zero D4-brane charge, we also expect an M5-brane charge in M-theory. However, we are interested in M-theory configurations in which the four-form is smooth and closed everywhere. There should be no M5-brane sources present, otherwise the membrane anomaly calculation, which will be our starting point for analysing the Freed-Witten anomaly, will not be valid.[3] In fact, one can see this problem when one tries to lift the flux G_4 to M-theory. Since $B = 0$, G is just the pull-back of G_4. In the limit $\epsilon \to 0$, so that the D4-brane charge is strictly confined to the D6-brane, the lift G of G_4 is singular at the locus where the M-theory circle vanishes. This is hardly surprising — we see from (2.9) that G_4 is also singular on Q. We

[3] It is an interesting open problem to study anomalies for membranes ending on M5-branes.

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could consider smoothing out the charge to a radius $\epsilon > 0$. This solves the singularity problem, but since we do not want an M5-brane charge in M-theory, this is not the right way to proceed.

2.2 Reduction of G and incorporation of the NS field

There is an obvious way to cancel the D4-brane charge, but still have non-trivial gauge field strength F. In general, F is replaced by the gauge-invariant quantity $(F - B)$ in the Wess-Zumino couplings on the D6-brane. Since B is a potential, it does not satisfy any quantisation condition, and we may simply choose B to cancel the flux F. Indeed, this is essentially what happens in the case of configurations which are dual to smooth M-theory solutions, as we now describe.

Let us begin with some M-theory configuration with flux G, equipped with a circle action with codimension four fixed point set Q. The U(1) action allows us to write

$$G = \tilde{G}_4 + H \wedge e_1 \,, \tag{2.10}$$

where $e_1 = (\mathrm{d}\psi - C_1)/2\pi$ denotes the global angular form on the M-theory circle bundle — that is, ψ is an angular coordinate on the M-theory circle and C_1 is the (pull-back of the) connection. The U(1) Killing vector is therefore $\partial/\partial\psi$. Since G is assumed to have no sources and $\mathcal{L}_{\partial/\partial\psi} G = 0$, where $\mathcal{L}$ denotes the Lie derivative, it follows that H is closed. Of course, this is just the statement that there are no NS5-brane sources present. From (2.10) H must also be zero on the locus Q where $\partial/\partial\psi$ vanishes in order that G be smooth there. This follows since e_1 is singular at Q — the dual Killing vector field vanishes there. The fact that G is closed then implies the Bianchi identity $\mathrm{d}\tilde{G}_4 = -H \wedge \frac{G_2}{2\pi}$. Notice that, although G_2 is ill-defined on Q, H vanishes on Q and therefore the Bianchi identity is in fact everywhere smooth.

On the other hand, if we incorporate the NS B-field in string theory, equation (2.8) is modified to read

$$\mathrm{d}\tilde{G}_4 = -H \wedge \frac{G_2}{2\pi} + \delta_Q \wedge (F - B) \,. \tag{2.11}$$

Here $\tilde{G}_4$ is the gauge-invariant four-form of type IIA, and is identified with what we called $\tilde{G}_4$ in M-theory. The first term in (2.11) arises from a bulk Chern-Simons coupling in type IIA supergravity. On the other hand, the last term in (2.11) arises by replacing F by the gauge-invariant quantity $F - B$ in the Wess-Zumino couplings on the D6-brane. Clearly, the two Bianchi identities for $\tilde{G}_4$ agree only if $F = B$ on the D6-brane. This is inevitable since, by assumption, $\tilde{G}_4$ in M-theory is everywhere smooth. In fact, all that we will need is that the cohomology classes of F and B on Q agree, $[F] = [B]$.

At this point, the reader may notice the following problem. The potential B may be computed from the G-flux in M-theory by $\mathrm{d}B = H$. In general this equation is only valid locally since the cohomology class of H may be non-trivial. Indeed, globally, (B, H) is really a Cheeger-Simons differential character, as we discuss at the end of this section. However, $\mathrm{d}B = H$ certainly holds in a tubular neighbourhood T of Q in Y since H actually vanishes on Q, and $H^3(T) \cong H^3(Q)$ since Q is a deformation retract of T. But then B is only uniquely defined modulo $B \to B + a$, where a is a closed two-form on Y, which defines a class $[a] \in H^2(Y;\mathbb{R})$. Thus $[F]_Q$, where the subscript emphasises that the cohomology is that of Q, is determined only modulo classes on Q that are the restrictions of cohomology classes on the whole of spacetime, Y. If $[a/2\pi] \in H^2(Y;\mathbb{Z})$ then $B \to B + a$ is a large gauge transformation of the B-field, and the corresponding ambiguity in F merely reflects the fact that it is $F - B$ which is the gauge-invariant quantity on Q. However, in general it seems that one must specify more precisely[4] the M-theory C-field in order to obtain $[F]$. As we will see, this ambiguity in $[F]$ will be irrelevant for deriving the Freed-Witten anomaly, and also the higher order Wess-Zumino terms on the D6-brane. This is just as well, since neither of these depends on the choice of C-field which satisfies $[\mathrm{d}C] = [G]$.

Consider now some closed U(1)-invariant four-dimensional submanifold $V \subset X$, with V having a codimension two fixed point set U on the locus Q. The situation of interest is when V describes a one-parameter family of membrane worldvolumes in spacetime, which descend to a family of open strings in type IIA with boundary $U \subset Q$. Consider the integral $\int_V G$. We compute

$$\int_V G = \int_{V/\,\mathrm{U}(1)} H\,, \tag{2.12}$$

where we have integrated over the M-theory circle. Notice that $\tilde{G}_4$ has no support over the M-theory circle, and therefore does not contribute to the integral. In the case that $H = \mathrm{d}B$ holds globally on Y we can use Stokes' theorem to write

$$\int_V G = \int_{V/\,\mathrm{U}(1)} H = \int_U B = \int_U F\,. \tag{2.13}$$

This implies the formula (1.3) that we were looking for. However, recall that $[F]_Q$ is only determined modulo $i^*[a]$, where $i : Q \to Y$ denotes the embedding map, and $[a] \in H^2(Y;\mathbb{R})$. However, for any such a, $\int_U a = 0$, since the homology class of U in Y is trivial. This follows because, by definition, U bounds $V/\,\mathrm{U}(1)$ in Y.

To complete the argument we must consider the case when $[H] \neq 0 \in H^3(Y)$. In fact, $[H/2\pi] \in H^3(Y;\mathbb{Z})$ as the NS field strength H is quantised. In this case, B cannot be a globally defined two-form — in fact it is more like a U(1) gauge field that has non-trivial first Chern class. Technically this means that (B,H) is a "Cheeger-Simons differential character". Concretely, this means that, due to the fact that B is not a globally defined object, the integral of B over a topologically trivial two-cycle $U \subset Y$ is only defined modulo 2π. Thus

$$\int_U B = \int_{V/\,\mathrm{U}(1)} H \qquad \mod 2\pi\,, \tag{2.14}$$

where recall that U is the boundary of $V/\,\mathrm{U}(1)$ in the case at hand. The idea here is that, if Z_1, Z_2 are any two 3-cycles with boundary $\partial Z_i = U$ $(i = 1, 2)$ then the integrals

$$\int_{Z_i} H \tag{2.15}$$

differ by $\int_Z H = 0 \mod 2\pi$, where Z is the *closed* three-manifold obtained by gluing Z_1 to Z_2 (with opposite orientation) along their common boundary U, so $Z = Z_1 \cup_U (-Z_2)$.

[4] I.e. not just the cohomology class of its curvature.

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Thus the definition

$$\int_U B = \int_{Z_i} H \quad \mod 2\pi \tag{2.16}$$

is well-defined — i.e. independent of the choice of $i = 1, 2$ — and of course is certainly true when H is exact. Thus more generally (2.13) holds modulo 2π. Thus, exponentiating everything (multiplied by i), we have proved (1.3).

In section 4 we will examine a concrete example and compute explicitly some of the quantities appearing in this section. In particular, H is topologically trivial on Y in this case, and there is a completely independent check on (2.13) from tadpole cancellation.

3. Fermion anomalies and a localisation formula

Our aim in this section is to give a proof of the "localisation formula" (1.4).

3.1 A review of the membrane anomaly

Consider the worldvolume theory for a membrane propagating on an oriented spin manifold X. Let W be a closed three-dimensional submanifold[5] of X, and consider wrapping the membrane on W. We focus on the following two terms in the membrane effective theory

$$\mathrm{Pf}(\mathcal{D}_W) \cdot \exp\left(i \int_W C\right) . \tag{3.1}$$

Here $\mathrm{Pf}(\mathcal{D}_W)$ denotes the pfaffian of the Dirac operator on the membrane worldvolume, and the second term in (3.1) is the "holonomy" of the M-theory C-field over W. In order to describe the pfaffian, let N_W denote the normal bundle of W in X. This has rank eight, and, since W and X are both spin, there exists a spin covering, which we denote $S(N_W) = S_+(N_W) \oplus S_-(N_W)$, and we have decomposed into positive and negative chirality. The membrane worldvolume fermions, after fixing kappa symmetry, are then spinors on W with values in $S_+(N_W)$ — also a real bundle of rank eight. Now, the spinors on W are pseudoreal. The Dirac operator is hermitean, and therefore its eigenvalues are real. For fermions valued in any real bundle, for example $S_+(N_W)$, the eigenvalues come in pairs — this is due to the existence of an anti-unitary symmetry in three dimensions. The fermion path integral is then given by the pfaffian of the Dirac operator valued in $S_+(NW)$, $\mathrm{Pf}(\mathcal{D}_W)$, and is roughly the square root of the determinant. Since the eigenvalues come in pairs, the determinant is formally positive, and regularisation preserves this property. The pfaffian is then formally given by

$$\mathrm{Pf}(\mathcal{D}_W) = \prod_n \lambda_n \,, \tag{3.2}$$

where the product runs over *pairs* of eigenvalues — that is, we include the contribution from one of the eigenvalues in each pair. The (regularised) pfaffian is then real, but there is no natural definition of its sign. This can potentially lead to an anomaly. Indeed, if one deforms the membrane around a one-parameter loop, the spectral flow in (3.2) might mean

[5] In fact, more generally we may allow W to be an immersion.

that the pfaffian changes sign as one traverses the loop. This spectral flow is given by the topological index theorem for families of Dirac operators. In order to describe this, notice that

$$N_W = \mathcal{O} \oplus N' \,, \tag{3.3}$$

where $\mathcal{O}$ is a trivial real line tangent to the $\mathbf{S}^1$ which parameterises the family of worldvolumes. Thus N' is a Spin(7) bundle. The number of eigenvalue pairs of the Dirac operator $\mathcal{D}_W$ that change sign in going around the circle (the spectral flow) is then given by [11]

$$\frac{1}{2}\text{Index } \mathcal{D}_{W\times\mathbf{S}^1} \,, \tag{3.4}$$

where $\mathcal{D}_{W\times\mathbf{S}^1}$ denotes the chiral Dirac operator on $W\times\mathbf{S}^1$, coupled to $S(N_{W\times\mathbf{S}^1})$. This operator arises by essentially gluing together the Dirac operators on $W\times\{q\}$, for each $q\in\mathbf{S}^1$, to make a Dirac operator on $W\times\mathbf{S}^1$. Note that, in four dimensions, the Dirac operator coupled to any real vector bundle has an index which is divisible by two and thus the above expression is indeed an integer. Using the index theorem one finds

$$\frac{1}{2}\text{Index } \mathcal{D}_{W\times\mathbf{S}^1} = \int_{W\times\mathbf{S}^1} \frac{p_1(X)}{2} \qquad \text{mod } 2\,. \tag{3.5}$$

In order to obtain this result one needs to note that all the characteristic classes of the tangent bundle of $W\times\mathbf{S}^1$ vanish.

To conclude, in order that the membrane quantum effective action be well-defined, the change in the sign of the pfaffian in (3.1) as one traverses the loop must be cancelled by the change in the holonomy factor $\int_W C$. This leads to the non-standard Dirac quantisation condition (1.2) for G.

3.2 A U(1)-index calculation

The idea in this section is simply to apply the U(1)-index theorem, where U(1) acts by rotating the M-theory circle fibres, to the Dirac operator $\mathcal{D}_{W\times\mathbf{S}^1}$, instead of the usual index theorem. This will lead directly to the result (1.4). We will therefore need to describe what the G-index theorem is.

It might be useful to first recall some of the details of the usual index theorem. We begin with a sequence $\{\mathcal{E}_k\}$ of smooth vector bundles over M, labelled by k, only finitely many of which are non-zero. The space of smooth sections of $\mathcal{E}_k$ is denoted $\Gamma(\mathcal{E}_k)$. We also assume we have differential operators $\mathcal{D}_k : \Gamma(\mathcal{E}_k)\to\Gamma(\mathcal{E}_{k+1})$ which form a *complex*: $\mathcal{D}_{k+1}\circ\mathcal{D}_k = 0$. If $\mathcal{D}_k^* : \Gamma(\mathcal{E}_{k+1})\to\Gamma(\mathcal{E}_k)$ is the dual operator, and $\Delta_k \equiv \mathcal{D}_k^*\mathcal{D}_k + \mathcal{D}_{k-1}\mathcal{D}_{k-1}^*$ is the laplacian, then the complex is said to be *elliptic* if the laplacian is an elliptic operator on $\Gamma(\mathcal{E}_k)$. We define the cohomology of the elliptic complex, in the usual way, to be

$$H^k(\mathcal{E},\mathcal{D}) = \frac{\ker(\mathcal{D}_k)}{\text{im}(\mathcal{D}_{k-1})} = \ker\Delta_k \,, \tag{3.6}$$

where the second equality follows from Hodge-de Rham theory, just as for the de Rham cohomology of manifolds — indeed the latter is just a special case of the above where $\mathcal{E}_k = \Lambda^k$ is the bundle of k-forms and the operators $\mathcal{D}_k$ are just the exterior derivative d, restricted to Λ^k. The *index* of the elliptic complex $(\mathcal{E}, \mathcal{D})$ is then defined to be

$$\mathrm{Index}(\mathcal{E}, \mathcal{D}) = \sum_k (-1)^k \dim H^k(\mathcal{E}, \mathcal{D})\,. \tag{3.7}$$

For example, for the exterior algebra this is just the Euler number of the base manifold M. The celebrated index theorem of Atiyah and Singer [20] relates the index, which is an analytic object, to certain characteristic classes integrated over M, which is a purely topological object.

Quite generally, we may also assume that we are given an action of the group G on our complex. Thus for each group element $g \in G$ we have smooth bundle maps $g_k : \mathcal{E}_k \to \mathcal{E}_k$ which commute with the $\mathcal{D}_k$ operators, $g_{k+1} \circ \mathcal{D}_k = \mathcal{D}_k \circ g_k$, and so lead to an induced action on the cohomology groups, $\hat{g}_k : H^k(\mathcal{E}, \mathcal{D}) \to H^k(\mathcal{E}, \mathcal{D})$. We may then define the *Lefschetz number* to be

$$L(g, \mathcal{E}, \mathcal{D}) = \sum_k (-1)^k \,\mathrm{tr}(\hat{g}_k)\,. \tag{3.8}$$

Clearly, if the action of g is trivial, this is just the usual index since the trace just computes the dimension of the cohomology groups. More generally we get a character of g. The G-index theorem is then a generalisation of the usual index theorem in which there is, in addition to the elliptic complex, a specified action of the group[6] $G : M \to M$, which "lifts" to an action on the complex $(\mathcal{E}, \mathcal{D})$ as described above [20]. The theorem then computes the Lefschetz number in terms of certain cohomology classes evaluated on the *fixed point set* M^g of g in M. Again, if the action of g is trivial, the theorem reduces to the usual index theorem.

To return to our problem, we have $G \cong \mathrm{U}(1)$ which acts on the loop of membrane worldvolumes $W \times \mathbf{S}^1$. This will be the base M. We then apply the U(1)-index theorem to the Dirac operator $\mathcal{D}_{W\times\mathbf{S}^1}$. This will give a formula for the Lefschetz number (3.8) in terms of certain characteristic classes evaluated on the fixed point set $U = \partial\Sigma \times \mathbf{S}^1$, which recall physically is a loop of string worldsheet boundaries on the D6-brane. Thus in the U(1)-index theorem we have $U = M^g$, for non-trivial $g \in \mathrm{U}(1)$. In order to utilise the theorem, we will need to work out how U(1) acts on the relevant bundle, which is essentially the spin bundle associated to the normal bundle $N_{W\times\mathbf{S}^1}$ of $W \times \mathbf{S}^1$ in X. This action is of course induced from the action of U(1) on the embedding space, X — the M-theory manifold in which the loop of membranes is sitting. As we will see explicitly below, the Lefschetz number is in fact *independent* of the particular group element $g \in \mathrm{U}(1)$ chosen, and is thus, setting g equal to the identity, equal to the usual index. The upshot is thus a formula for the usual index in terms of certain cohomology classes evaluated on the fixed point set of the circle action. Of course, this is precisely the form of (1.4). Before we begin with the details, we suggest here that the interested reader might consult [19, section 7.6] for further background on G-index theorems, as well as some simpler examples.

[6] There are some additional technical assumptions that we are suppressing for the time being. For example, G must be "topologically cyclic", which simply means that there is a group element $g \in G$ whose powers are dense in G.

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We denote the bundle $S(N_{W\times\mathbf{S}^1})$ as E, and the Dirac operator on $W\times\mathbf{S}^1$ coupled to E as $\mathcal{D}$, and of course $W\times\mathbf{S}^1$ is M. Thus we have a two-term complex $\mathcal{E}=S^+\otimes E, S^-\otimes E$, where $S^\pm$ are the chiral spinor bundles of the four-manifold $M=W\times\mathbf{S}^1$. We are now in line with the notation above. The membrane anomaly is computed from $\mathrm{Index}(\mathcal{E},\mathcal{D})$, and we would now like to compute the Lefschetz number by applying the G-index theorem.

It is a relatively straightforward exercise to write down the form of the G-index theorem for a Dirac operator coupled to a vector bundle $E\to M$ with G-action, starting from the general theorem in [20]. The result is

$$L(g,\mathcal{E},\mathcal{D})=(-1)^l\int_{M^g}\prod_j\left(-2i\sin\frac{\theta_j}{2}\right)^{-s_j}\hat{A}(M^g)\ \prod_j\mathcal{M}^{\theta_j}(N^g(\theta_j))\ \mathrm{ch}(E\,|_{M^g})(g)\,. \quad (3.9)$$

This rather formidable expression is fairly straightforward to explain. M^g denotes the fixed point set of g, which has dimension $2l=\dim M^g$, and N^g is its normal bundle in M. Now, since g fixes M^g we get an induced action on its normal bundle N^g in M, which then induces a splitting of N^g into a sum complex vector bundles and real vector bundles. This follows from simple representation theory of cyclic groups — recall that G is required to be topologically cyclic. In fact we have omitted the contribution from the real vector bundle in (3.9) since it is not present for a generic element of U(1) — for $G\cong\mathrm{U}(1)$, we already remarked earlier in section 2 that we obtain an induced complex structure on the normal bundle. Thus we have a bundle decomposition

$$N^g=\bigoplus_j N^g(\theta_j)\,, \quad (3.10)$$

where the action of g on the complex vector bundle $N^g(\theta_j)$ is by definition multiplication by $e^{i\theta_j}$. Moreover, the complex dimension of this bundle is s_j, so that $\sum_j s_j=s=\dim_{\mathbb{C}}N^g$.

Finally, $\hat{A}$ is the usual Dirac genus,

$$\hat{A}=1-\frac{p_1}{24}+\frac{7p_1^2-4p_2}{5760}+\cdots \quad (3.11)$$

$\mathcal{M}^\theta$ is the (stable) characteristic class given by the formula[7]

$$\mathcal{M}^\theta=1+\frac{i}{2}\cot\frac{\theta}{2}c_1+\cdots\,, \quad (3.12)$$

where the dots denote higher order terms which we will not need, and $\mathrm{ch}(E\,|_{M^g})(g)$ denotes the equivariant Chern character of the bundle E restricted to the fixed point set M^g. This Chern character replaces the usual Chern character in the index theorem, and is in fact the most important term for us. We will therefore need to describe this object.

Recall that, in the usual index theorem, one encounters the Chern character $\mathrm{ch}(E)\in H^*(M;\mathbb{Q})$. For example, for a spin bundle E associated to a vector bundle N — the case of interest here — one has the formula

$$\mathrm{ch}(E)=\prod_k\left(e^{y_k/2}+e^{-y_k/2}\right). \quad (3.13)$$

[7]This formula was also used in [8].

where the y_k are the basic characters of N. If one "imagines" that N is in fact a direct sum of complex line bundles, then the y_k are just the first Chern classes of these line bundles. This is known as the splitting principle.

In the equivariant case we encounter the object $\mathrm{ch}(E\mid_{M^g})(g)$. The G-index theorem naturally involves *equivariant* K-theory, which is where this object comes from. Here we simply sketch the ideas and state the formula — for further details the inquisitive reader should consult the literature.

Consider the situation where the group G acts trivially on a manifold Z, which for us will be the fixed point set M^g in the G-index theorem. Then a basic result is that the equivariant K-theory of Z is simply the tensor product

$$K_G(Z) \cong K(Z) \otimes R(G)\,, \tag{3.14}$$

where $R(G)$ denotes the character ring of G. Suppose then that we have an element $u = x \otimes \chi \in K_G(Z)$ where $x \in K(Z)$, and χ is a character of G. Then the equivariant Chern character [20] is a map from $K_G(Z) \to H^*(Z;\mathbb{C})$ defined as follows:

$$\mathrm{ch}(u)(g) = \chi(g)\cdot \mathrm{ch}(x) \in H^*(Z;\mathbb{C})\,, \tag{3.15}$$

where $\mathrm{ch}(x)$ is the usual Chern character. In the G-index theorem we encounter the complex vector bundles $N^g(\theta_j)$ where, by definition, the action of $g \in G$ on $N^g(\theta_j)$ is multiplication by $e^{i\theta_j}$. If y_k denote the basic characters of $N^g(\theta_j)$, $k = 1,\dots,s_j = \dim_{\mathbb{C}} N^g(\theta_j)$, we have, for fixed j, the contribution

$$\mathrm{ch}(u)(g) = e^{i\theta_j}\cdot \mathrm{ch}(N^g(\theta_j)) = \sum_k e^{i\theta_j}\cdot e^{y_k} = \sum_k e^{y_k+i\theta_j}\,. \tag{3.16}$$

Thus, in the equivariant Chern character, one simply replaces the basic characters y_k of the j'th complex vector bundle by $y_k + i\theta_j$ in the formula for the usual Chern character. We will use this fact below.

We now specialise to the case $G \cong \mathrm{U}(1)$. The fixed point set is $M^g = U = \partial\Sigma \times \mathbf{S}^1$, and so $l = 1$, and the normal bundle of U in $M = V = W \times \mathbf{S}^1$ is just a complex line, so that $s = 1$. Indeed, notice that the normal bundle to Q in X, which we denote by $\mathcal{V}$, has real rank four. $\mathcal{V}$ therefore splits into the sum of two complex line bundles under the $\mathrm{U}(1)$ action when restricted to U, $\mathcal{V}\mid_U = \mathcal{L}_1 \oplus \mathcal{L}_2$, with $\mathrm{U}(1)$ acting as multiplication by $e^{i\theta}$ on each factor — here $0 \le \theta \le 2\pi$ is now literally the $\mathrm{U}(1)$ group parameter $g = e^{i\theta}$. We may take $\mathcal{L}_1$ to be normal to $\partial\Sigma \times \mathbf{S}^1$ in $W \times \mathbf{S}^1$ — that is, $\mathcal{L}_1$ is just the normal bundle N^g of M^g in M. Thus $N^g = N^g(\theta)$, in the above notation.

Evaluating (3.9) we obtain

$$L(\theta,\mathcal{E},\mathcal{D}) = -\frac{i}{2}\mathrm{cosec}\frac{\theta}{2}\int_U \left[1 + \frac{i}{2}\cot\frac{\theta}{2}c_1(\mathcal{L}_1) + \cdots\right]\ \mathrm{ch}(E\mid_U)(g)\,. \tag{3.17}$$

It remains to compute the equivariant Chern character. Recall that E is the spin bundle for N_M, where $M = W \times \mathbf{S}^1$. N_M is a rank seven vector bundle (since $7 = 11 - 4$), and over the fixed point set U we obtain a splitting

$$N_M\mid_U = \mathcal{F} \oplus \mathcal{L}_2\,, \tag{3.18}$$

where $\mathcal{L}_2$ is the line bundle which appears in the decomposition $\mathcal{V}\mid_U = \mathcal{L}_1 \oplus \mathcal{L}_2$. The group U(1) acts trivially on the real rank five bundle $\mathcal{F}$ — which corresponds to the directions in the D6-brane transverse to the loop of string boundary — but rotates the line bundle $\mathcal{L}_2$ by the action $e^{i\theta}$. Let us denote the basic characters for $\mathcal{F}$ as y_k, $k = 1, 2$, and let the first Chern class of $\mathcal{L}_2$ be denoted $y = c_1(\mathcal{L}_2)$. Recalling that E is the spin bundle associated to N_M, we may therefore compute the equivariant Chern character using (3.13):

$$\mathrm{ch}(E\mid_U)(\theta) = \prod_k \left(e^{y_k/2} + e^{-y_k/2}\right) \cdot \left(e^{(y+i\theta)/2} + e^{-(y+i\theta)/2}\right). \tag{3.19}$$

This follows from our above discussion, where we argued that the Chern class y of $\mathcal{L}_2$ gets replaced by $y + i\theta$ in the equivariant formula; $\mathcal{F}$ has a trivial U(1) action, and so the $\theta_k = 0$ for this bundle.

On substituting this expression into (3.17) and using standard trigonometric formulae, one arrives at the result

$$L(\theta, \mathcal{E}, \mathcal{D}) = \int_U 2c_1(\mathcal{L}_2) + 2\cot^2\frac{\theta}{2} c_1(\mathcal{L}_1)\,. \tag{3.20}$$

Now, in fact

$$\int_U c_1(\mathcal{L}_1) = 0\,. \tag{3.21}$$

This is a simple consequence of the fact that $\mathcal{L}_1$ is the normal bundle of $M^g = U$ in $M = V$, and the circle action which rotates $\mathcal{L}_1$ extends over $M = V$ without any other fixed points. Specifically, we have

$$\int_U c_1(\mathcal{L}_1) = \int_{V/\,\mathrm{U}(1)} \mathrm{d}\left[c_1(\mathcal{L}_1)\right] = 0\,, \tag{3.22}$$

where $V/\,\mathrm{U}(1)$ is the quotient three-manifold, which has boundary U, and we have simply used Stokes' theorem. In general, $M^g = U$ need not be connected and then, in principle, $c_1(\mathcal{L}_1)$ could be non-zero on some components, as long as (3.21) holds. Thus we see that the Lefschetz number is actually *independent* of $g = e^{i\theta}$. But setting $g = 1$ in (3.8) we of course obtain the usual index. Thus we have shown that

$$\frac{1}{2}\mathrm{Index}\ \mathcal{D} = \int_U c_1(\mathcal{L}_2)\,. \tag{3.23}$$

Now, we have $y = c_1(\mathcal{V})$, restricted to U, where we have used (3.21). But then $c_1(\mathcal{V}) = w_2(\mathcal{V})$ modulo two, and since X is oriented and spin it follows that $c_1(\mathcal{V}) = w_2(Q)$ modulo two also. Thus

$$\frac{1}{2}\mathrm{Index}\ \mathcal{D} = \int_U w_2(Q) \mod 2\,. \tag{3.24}$$

Combining this with the usual index theorem (3.5) gives (1.4).

4. Wess-Zumino couplings and an example

In this section we consider a concrete example where the anomalies in question are non-trivial, and also discuss the Wess-Zumino terms on the D6-brane. There is then an independent check of some of our results which arises by considering tadpole cancellation. This also ties in naturally with reference [8].

Consider M-theory on $\mathbb{R}^{1,2} \times \mathbb{HP}^2$, where $\mathbb{HP}^2$ denotes quaternionic projective two-space. The isometry group of this space is $Sp(3)$ and there is an embedding $\mathrm{U}(3) \subset Sp(3)$, which amounts to the embedding $\mathbb{C} \subset \mathbb{H}$. Then the action of the diagonal $\mathrm{U}(1) \subset \mathrm{U}(3)$ on[8] $X = \mathbb{HP}^2$ has a fixed point set $\mathbb{CP}^2$ [2, 4]. In fact, the generic orbit under the $\mathrm{U}(3)$ action is a copy of the Aloff-Wallach space $N_{1,-1} = \mathrm{SU}(3)/\mathrm{U}(1)$. This has codimension one in X. There is then a theorem that we may apply which which states that there are then precisely two "special orbits" of higher codimension. In the case at hand, one of these is a copy of $\mathbf{S}^5$, which is Hopf-fibred over $\mathbb{CP}^2$ by the circle action, and the other is a copy of $\mathbb{CP}^2$, which is left fixed by $\mathrm{U}(1)$. The latter is thus a codimension four fixed point set and so will become our D6-brane worldvolume Q. A full discussion of this orbit structure may be found in [2].

Since $\mathbb{CP}^2$ is not spin, the gauge field strength on a D6-brane wrapped on $\mathbb{R}^{1,2} \times \mathbb{CP}^2$ has periods which are half-integer multiples of 2π. In fact, there is only one non-trivial two-cycle $U = \mathbb{CP}^1 \subset \mathbb{CP}^2$, and so we may generally write

$$\int_{\mathbb{CP}^1} \frac{F}{2\pi} = \frac{1}{2} + n \tag{4.1}$$

for some $n \in \mathbb{Z}$. This integer completely characterises the flux in this case.

It is also easy to analyse the membrane anomaly on $X = \mathbb{HP}^2$. The integral cohomology $H^4(\mathbb{HP}^2;\mathbb{Z})$ is generated by a four-form λ. In fact it is quite straightforward to show that λ is precisely half the first Pontryagin class of $\mathbb{HP}^2$. It follows of course that λ is not divisible by two, and so the membrane anomaly is non-trivial — indeed, from the last section we know that this must be the case. As for the complex projective space, $\mathbb{HP}^2$ has only one non-trivial cycle — dual to λ — which is a linearly embedded $\mathbb{HP}^1 \cong \mathbf{S}^4$. In general we may therefore write

$$\int_{\mathbb{HP}^1} \frac{G}{2\pi} = \frac{1}{2} + m \tag{4.2}$$

for some $m \in \mathbb{Z}$, which again completely characterises the four-form flux.

Now, the cycle $\mathbb{HP}^1$ is acted on by the circle action, with fixed point set being the non-trivial $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Indeed, notice that the circle action on $\mathbb{HP}^1$ *must* have fixed points somewhere since the Euler number of $\mathbb{HP}^1$ is 2. The Lefschetz fixed point formula then asserts that the Euler number is the sum of the Euler numbers of the fixed point sets — this is in fact a simple application of the G-index theorem where one uses the de Rham complex. Thus there are only two obvious possibilities — either one fixes two points (the north and south poles of the four-sphere), or else one fixes a copy of $\mathbf{S}^2$. In fact it is the latter that is the case here. For further details the reader is referred to [4]. Moreover, the quotient space $Y = X/\mathrm{U}(1)$ is extremely simple — it is the seven-sphere $\mathbf{S}^7$. This non-obvious fact is proved in [2]. In particular, notice that H is necessarily cohomologically trivial since $H^3(\mathbf{S}^7;\mathbb{Z}) = 0$ and thus our formula (2.13) holds and gives $n = m$. We next proceed to show how one may use tadpole cancellation to check this. However, before doing this, it might be instructive to explicitly compute the terms in the G-index theorem for this space.

[8] In this section the $\mathbb{R}^{1,2}$ factor does not play any role in the discussion, and so we project it out of our formulae. Thus X will denote the non-trivial part of the M-theory spacetime. Similar remarks will apply elsewhere.

The index of the Dirac operator on $\mathbf{S}^4 \subset X$ coupled to the normal spin bundle gives

$$\frac{1}{2}\text{Index } \mathcal{D} = \int_{\mathbf{S}^4} \lambda = 1\,, \tag{4.3}$$

where the last step follows since λ is dual to the four-cycle $\mathbf{S}^4$, as discussed above. According to our U(1)-index calculation (3.23) this should equal

$$\int_{\mathbf{S}^2} c_1(\mathcal{V})\,, \tag{4.4}$$

where recall that $\mathcal{V}$ is the normal bundle of $\mathbb{CP}^2$ in X — i.e. the normal bundle of the fixed point set. Thus we need to know the normal bundle of $\mathbb{CP}^2$ in $\mathbb{HP}^2$. In fact, as shown[9] in [4], this is the "universal quotient bundle" of $\mathbb{CP}^2$. This may be defined as follows. One begins with the trivial bundle over $\mathbb{CP}^2$ of complex rank 3, which is thus simply the product $\mathbb{CP}^2 \times \mathbb{C}^3$. There is then a natural complex line bundle $\mathcal{S}$ over $\mathbb{CP}^2$ which may be defined as the subbundle consisting of pairs $(p, l) \in \mathbb{CP}^2 \times \mathbb{C}^3$ where l is the complex line in $\mathbb{C}^3$ corresponding to the point $p \in \mathbb{CP}^2$. The universal quotient bundle is then simply the orthogonal complement of $\mathcal{S}$ in $\mathbb{CP}^2 \times \mathbb{C}^3$. From this definition one easily sees that the first Chern class of this rank two complex vector bundle is indeed equal to the generator of $H^2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$, which is dual to the two-cycle $\mathbf{S}^2$, so that the integral (4.4) is 1. Thus we have verified the result of our general U(1)-index calculation in this explicit example.

4.1 Wess-Zumino terms and tadpole cancellation

As reviewed in [8], tadpole cancellation for the C-field yields

$$N_{M2} + \frac{1}{192}\int_X (p_1^2 - 4p_2) + \frac{1}{2}\int_X \left(\frac{G}{2\pi}\right)^2 = 0\,. \tag{4.5}$$

Here N_{M2} is the number of space-filling M2-branes in $\mathbb{R}^{1,2}$, located at some points in X, and p_i denote Pontryagin classes. The latter arise from the gravitational correction to the eleven-dimensional supergravity action mentioned in the introduction. The last term in (4.5) comes from the usual Chern-Simons term in eleven-dimensional supergravity.

There is a similar tadpole condition for C_3 in type IIA which arises due to the Wess-Zumino terms on a D6-brane [8] wrapped on $\mathbb{R}^{1,2} \times Q$. In general there is also a bulk contribution. The tadpole condition then reads

$$N_{D2} + \int_Q \sqrt{\hat{A}(TQ)/\hat{A}(NQ)} + \frac{1}{2}\int_Q \left(\frac{F-B}{2\pi}\right)^2 + \int_Y \frac{\tilde{G}_4}{2\pi} \wedge \frac{H}{2\pi} = 0\,. \tag{4.6}$$

Here N_{D2} is the number of space-filling D2-branes, which is clearly identified with N_{M2} in M-theory.

The main result of [8] was that, for general X and D6-brane configuration Q, the gravitational terms in (4.5) and (4.6) are equal. Using the results of section 2, one can verify that the remaining Wess-Zumino and bulk terms in (4.6) descend from the G-flux

[9] This example is closely related to certain Spin(7) manifolds.

terms in (4.5). Indeed, the Wess-Zumino term involving $F - B$ of course vanishes. Since the bulk term in (4.6) is simply the dimensional reduction of the flux term in (4.5), the equivalence of the tadpole conditions is then clear.

However, one may now integrate the bulk term in (4.6) by parts[10] and, using the Bianchi identity for $\tilde{G}_4$ and fact that $F = B$ on Q, one finds that the bulk term may be written as

$$\frac{1}{2}\int_Q \left(\frac{F}{2\pi}\right)^2 . \tag{4.7}$$

Thus the bulk term mimics a Wess-Zumino coupling on the brane in which $B = 0$.

It is easy to check explicitly in the case of our example that the various terms in the tadpole cancellation conditions match up. One needs the following topological information: $p_1(\mathbb{HP}^2) = 2\lambda$, $p_2(\mathbb{HP}^2) = 7$, $p_1(\mathbb{CP}^2) = 3$, $p_1(\Lambda^-\mathbb{CP}^2) = -3$. Here $\Lambda^-\mathbb{CP}^2$ denotes the bundle of anti-self-dual two-forms over $\mathbb{CP}^2$, which is the normal bundle of $\mathbb{CP}^2$ in $Y = \mathbf{S}^7$. The result for the first Pontryagin class follows easily since the embedding space is a seven-sphere. It was also computed explicitly in [3, appendix A]. Substituting these values into the above formulae, one finds that the gravitational terms are both equal to $-1/8$. Thus

$$N_{M2} = -\frac{1}{2}m(m+1) \tag{4.8}$$

and

$$N_{D2} = -\frac{1}{2}n(n+1)\,. \tag{4.9}$$

Identifying[11] $N_{M2} = N_{D2}$, we therefore find that either $n = m$, or $n = -1 - m$ — the two choices simply correspond to opposite signs for the flux. Of course, we have shown in a completely different way that $n = m$, and so we must pick the first solution. This is therefore an independent check on (2.13).

5. Freed-Witten anomaly from K-theory

For completeness, in this section we show how one may also derive the shift (1.1) in the periods of F by using the K-theory formula [21, 22] for the Ramond-Ramond four-form G_4. For simplicity, we set the NS field to zero for the rest of the paper.

The starting point is the relation

$$\int_{\mathbf{S}^2} \frac{G_4}{2\pi} = \frac{F}{2\pi}\,, \tag{5.1}$$

where recall that $\mathbf{S}^2$ is any two-sphere linking the D6-brane worldvolume Q. Now, if G_4 had periods that were multiples of 2π, then one would conclude from (5.1) that F is also standard Dirac quantised. However, Ramond-Ramond fields are more properly interpreted in K-theory [21]. Roughly, G_4 — away from the D6-brane — is given by the four-form

[10] For simplicity we assume that H is topologically trivial. In particular this is the case in our example.

[11] Of course, N_{M2} is allowed to be negative, corresponding to a non-zero number of space-filling anti-M2-branes.

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piece of the Chern character of a K-theory class $x \in K(Y \setminus Q)$. Here $Y \setminus Q$ denotes Y with the D6-brane Q deleted. More precisely, we have the following quantisation condition on G_4:

$$\left[\frac{G_4}{2\pi}\right] = \left[\sqrt{\hat{A}(Y \setminus Q)} \cdot \mathrm{ch}\left(x + \frac{\Theta}{2}\right)\right]_{\mathrm{four-form}} \in H^4(Y \setminus Q; \mathbb{Q})\,. \tag{5.2}$$

We will not need to know much about the class Θ. We simply note that, expanding (5.2), we obtain [22]

$$\left[\frac{G_4}{2\pi}\right] = \frac{1}{2}c_1^2(x) - c_2(x) - \frac{1}{2}\lambda(Y \setminus Q) \qquad \mathrm{mod}\ \mathbb{Z}\,, \tag{5.3}$$

where $\lambda = p_1/2$. Now, in a tubular neighbourhood of the D6-brane, this class is a pull-back from Q, and so, although at first sight its contribution in (5.3) may appear half-integral, cannot contribute to (5.1). In fact, one can show[12] that λ is even in such a tubular neighbourhood. Thus it must be the first factor in (5.3) which leads to the half-integer shifts in the periods of F. Indeed, $c_1(x)$ is the first Chern class of the M-theory circle bundle, and thus may be identified with the cohomology class of $G_2/2\pi$. Thus we want to compute

$$\int_{\mathbf{S}^2}\left[\frac{G_2}{2\pi}\right]^2 \qquad \mathrm{mod}\ 2\,. \tag{5.4}$$

There are several ways of doing this. One way is to use an argument similar[13] to the one in [12, section 5.3]. One finds that (5.4) is given by $w_2(NQ)$ modulo 2, and therefore $w_2(Q)$ modulo 2. This is the result we were looking for. However, in this particular case we can do rather better. Let SQ denote the total space of the normal sphere bundle to Q (in other words, the boundary of a tubular neighbourhood, T). Then one knows the cohomology ring of SQ in terms of that of Q — it is given by a polynomial ring

$$H^*(SQ) \cong H^*(Q)[z]/\left(z^2 - c_1(\mathcal{V})z + c_2(\mathcal{V})\right)\,, \tag{5.5}$$

where recall that $\mathcal{V}$ is the normal bundle to Q in X, viewed as a complex rank two vector bundle. Here z, which generates the cohomology of the sphere fibres, may be identified with $c_1(x)$. This formula follows since SQ is the projectivisation of $\mathcal{V}$ — see [14, equation (20.7)]. Thus we can compute

$$\int_{\mathbf{S}^2} z^2 = -\int_{\mathbf{S}^2}(-c_1(\mathcal{V})z + c_2(\mathcal{V})) = c_1(\mathcal{V})\,. \tag{5.6}$$

Finally, recall that $c_1(\mathcal{V})$ reduces to $w_2(Q)$, modulo 2.

6. Conclusions and some speculative remarks

By interpreting codimension four fixed point sets in M-theory as D6-branes in type IIA, and codimension two fixed point sets on membranes as string boundaries, we have succeeded in deriving the global worldsheet anomaly for strings ending on a D6-brane, starting from

[12] To see this one needs to note that the fourth Wu class of Q vanishes on dimensional grounds.

[13] Here the computation was for $[G/2\pi]^2$ over a four-sphere linking an M5-brane worldvolume.

M-theory. Together with the results of [8], we have also shown that the Wess-Zumino terms on a D6-brane may be derived from the Chern-Simons terms in M-theory. It is amusing to consider[14] the case of fixed point sets which have a different (co)dimension. For example, can one make sense of a codimension two fixed point set? This would naturally become a single-sided boundary in type IIA. Moreover, strings would appear to be able to end on such an object. It is tempting to intepret this as some sort of single-sided D8-brane, although there are many problems with this interpretation. And what about higher codimension? One runs into an immediate problem for codimension six, since then taking the projectivisation down to type IIA gives $\mathbb{CP}^2$ as fibre. In this case it is not clear how to interpret the fixed point set in type IIA. Indeed, since $\mathbb{CP}^2$ does not bound, one even has problems defining the type IIA manifold.

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